Rotation (Pre Lecture)

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Introduction

Rotations

- You've dealt with planar rotations in past (math/physics/engineering) classes
  1. Angles, sin, and cos
  2. Complex (imaginary) numbers
- Three-dimensional rotation is more complex than planar rotation

Outcomes

- Apply complex numbers to describe planar rotations (review)
- Apply **quaternions** to describe 3D rotations
- Relate quaternion/polar and matrix/rectangular rotation representations

Outline

The "Imaginary" Number

$$\hat{\imath}^2 = -1$$

$$c = \overbrace{\underbrace{a}_{\text{real}} + \underbrace{b\hat{\imath}}_{\text{imaginary}}}^{\text{complex number}}$$

Complex Algebra

$$a = a_r + a_i\hat{\imath}$$

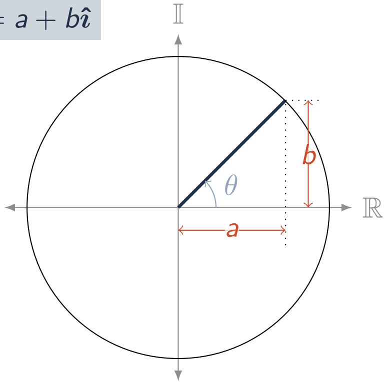
$$b = b_r + b_i\hat{\imath}$$

Addition: $$(a_r + a_i\hat{\imath}) + (b_r + b_i\hat{\imath}) = (a_r + b_r) + (a_i + b_i)\hat{\imath}$$

Multiplication: $$(a_r + a_i\hat{\imath})(b_r + b_i\hat{\imath}) = (a_r b_r - a_i b_i) + (a_r b_i + a_i b_r)\hat{\imath}$$

Complex Plane

$$c = a + b\hat{\imath}$$



Euler's Formula

Theorem: Euler's Formula

$$e^{\theta \hat{\imath}} = \cos\theta + \hat{\imath}\sin\theta$$

- Exponential Properties:

  zero: $e^0 = 1$

  derivative: $f^{(n)}\left(e^x\right) = e^x$

- Maclaurin Series: $f(x) = \frac{f(0)}{0!}\left(x^0\right) + \frac{f'(0)}{1!}\left(x^1\right) + \frac{f''(0)}{2!}\left(x^2\right) + \ldots$
  - $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$
  - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$
  - $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$
- $\hat{\imath}^0 = 1 \quad \hat{\imath}^1 = \hat{\imath} \quad \hat{\imath}^2 = -1 \quad \hat{\imath}^3 = -\hat{\imath}$

  $\hat{\imath}^4 = 1 \quad \ldots$

Euler's Formula

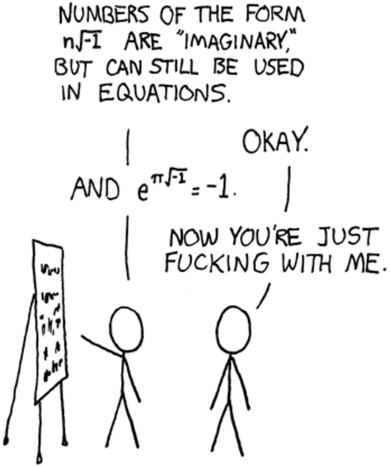
Proof

Proof Outline

$$
\begin{aligned}
e^{\theta \hat{\imath}} &= \frac{1}{0!} + \frac{(\theta \hat{\imath})^1}{1!} + \frac{(\theta \hat{\imath})^2}{2!} + \frac{(\theta \hat{\imath})^3}{3!} + \frac{(\theta \hat{\imath})^4}{4!} + \frac{(\theta \hat{\imath})^5}{5!} + \ldots \\
&= \frac{1}{0!} + \frac{\theta \hat{\imath}}{1!} - \frac{\theta^2}{2!} - \frac{\theta^3 \hat{\imath}}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5 \hat{\imath}}{5!} + \ldots \\
&= \left(\frac{1}{0!} - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots \right) + \left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots \right) \hat{\imath} \\
&= \cos\theta + \hat{\imath} \sin\theta
\end{aligned}
$$

□

Exercise: Euler's Formula



https://xkcd.com/179/

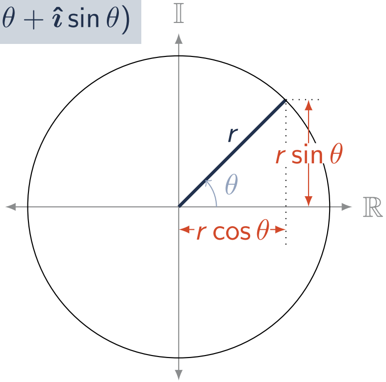
Proof

1. $e^{\pi\sqrt{-1}}$

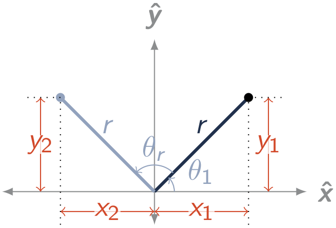
Complex Logarithm

1. $e^{\hat{\imath}\theta} = \cos\theta + \hat{\imath}\sin\theta$
2. $re^{\hat{\imath}\theta} = r\left(\cos\theta + \hat{\imath}\sin\theta\right)$
3. $e^{\ln r}e^{\hat{\imath}\theta} = r\left(\cos\theta + \hat{\imath}\sin\theta\right)$
4. $e^{\ln r + \hat{\imath}\theta} = r\cos\theta + \hat{\imath}r\sin\theta$
5. $\ln r + \hat{\imath}\theta = \ln\left(r\cos\theta + \hat{\imath}r\sin\theta\right)$
   - $a = r\cos\theta$
   - $b = r\sin\theta$
6. $\ln\left(a + b\hat{\imath}\right) = \ln r + \hat{\imath}\theta$, where
   - $\theta = \operatorname{atan2}\left(b, a\right)$
   - $r = \sqrt{a^2 + b^2}$

$re^{\theta\hat{\imath}} = r(\cos\theta + \hat{\imath}\sin\theta)$



Planar Rotations



Rectangular / Matrix	Polar / Complex
$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = r \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix}$	$re^{\theta_1 \hat{\imath}} = x_1 + y_1 \hat{\imath}$
$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = r \begin{bmatrix} \cos(\theta_1 + \theta_r) \\ \sin(\theta_1 + \theta_r) \end{bmatrix}$	$re^{(\theta_1 + \theta_r)\hat{\imath}} = x_2 + y_2 \hat{\imath}$
$= r \begin{bmatrix} \cos\theta_1 \cos\theta_r - \sin\theta_1 \sin\theta_r \\ \cos\theta_1 \sin\theta_r + \sin\theta_1 \cos\theta_r \end{bmatrix}$	
$= \begin{bmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & +\cos\theta_r \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$	$\left(re^{\theta_1 \hat{\imath}}\right) e^{\theta_r \hat{\imath}} = x_2 + y_2 \hat{\imath}$

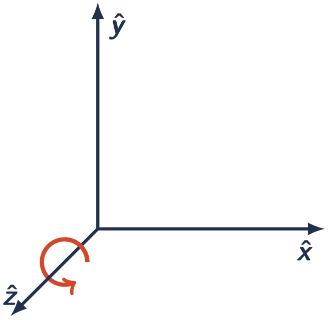
Computational Issues

- Sounds complicated. Why not just use angles, sin, and cos?
  1. Efficiency: sin/cos are expensive to compute.
     Multiplies and adds (matrix/complex) are cheaper
  2. Generalization to 3D
- Floating Point Error:
  - $\begin{bmatrix}\cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & +\cos\theta_1\end{bmatrix} \overset{\texttt{fp}}{*} \begin{bmatrix}\cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & +\cos\theta_2\end{bmatrix} = \begin{bmatrix}c_1c_2 - s_1s_2 & -c_1s_2 - s_1c_2 \\ c_1s_2 + s_1c_2 & c_1c_2 - s_1s_2\end{bmatrix} + \begin{bmatrix}e_{11} & e_{12} \\ e_{21} & e_{22}\end{bmatrix}$
  - Normalize Rotation Matrix: Gram-Schmidt process
  - $(\cos\theta_1 + \hat{\imath}\sin\theta_1) \overset{\texttt{fp}}{*} (\cos\theta_2 + \hat{\imath}\sin\theta_2) = (c_1c_2 - s_1s_2) + (c_1s_2 + s_1c_2)\,\hat{\imath} + e_r + \hat{\imath}e_i = \tilde{c} + \hat{\imath}\tilde{s}$
  - Normalize complex: $\tilde{c} + \hat{\imath}\tilde{s} \quad \rightsquigarrow \quad \frac{\tilde{c}+\hat{\imath}\tilde{s}}{\sqrt{\tilde{c}^2+\tilde{s}^2}}$

Outline

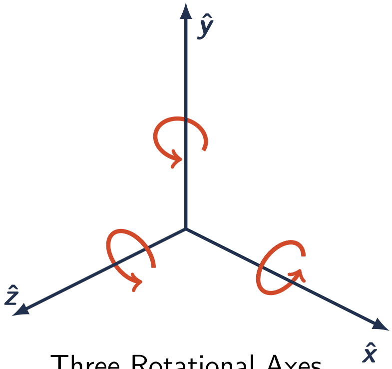
Geometric Intuition

2D



One Rotational Axis

3D



Three Rotational Axes

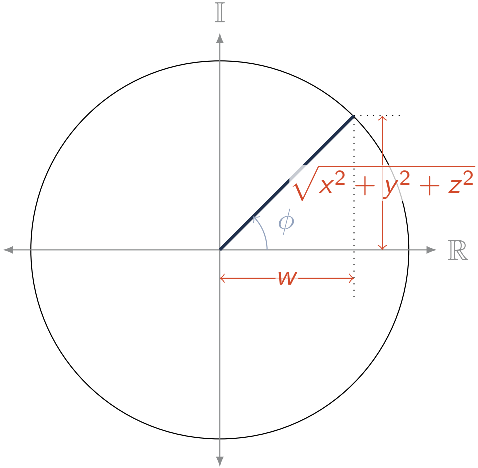
The Quaternion Axiom

$$\hat{\boldsymbol{\imath}}^2 = \hat{\boldsymbol{\jmath}}^2 = \hat{\boldsymbol{k}}^2 = \hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}\hat{\boldsymbol{k}} = -1$$

Quaternions

$$\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = \hat{\imath}\hat{\jmath}\hat{k} = -1$$

$$h = \overbrace{\underbrace{w}_{\text{scalar/real}} + \underbrace{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}_{\text{vector/imaginary}}}^{\text{quaternion}}$$



Why not use three angles?

Conventions: 12 varying axis + 12 fixed axis:

Varying Axis: xyz, xzy, yxz, yzx, zyx, zxy, xyx, xzx, yxy, yzy, zxz, zyz

Fixed-Axis: xyz, xzy, yxz, yzx, zyx, zxy, xyx, xzx, yxy, yzy, zxz, zyz

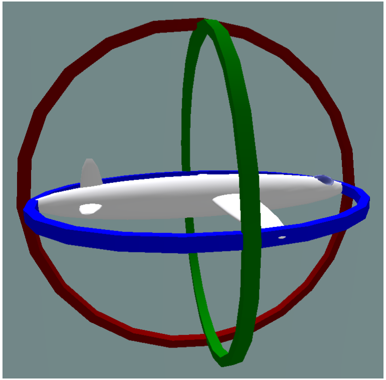
Sequential Rotations: Sequence of rotations is no longer addition of angles. Axes change with each rotation.

Singularities: Aligned axes can remove a degree-of-freedom ("gimbal lock")

Euler Angle Singularities

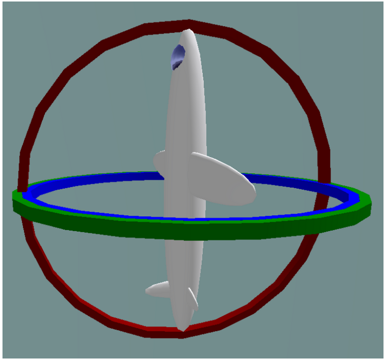
Gimbal Lock

OK



https://commons.wikimedia.org/wiki/File:No_gimbal_lock.png

Singularity



https://commons.wikimedia.org/wiki/File:Gimbal_lock.png

The Quaternion Axiom

$$\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = \hat{\imath}\hat{\jmath}\hat{k} = -1$$

$$h = \underbrace{w}_{\text{scalar}} + \underbrace{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}_{\text{vector}}$$

Example: Multiplication of Quaternion Units

$$\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}} = ?$$

0.	$\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}\hat{\boldsymbol{k}}$	$=$	-1	start with quaternion axiom
1.	$\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}\hat{\boldsymbol{k}}^2$	$=$	$-\hat{\boldsymbol{k}}$	post-multiply by $\hat{\boldsymbol{k}}$
2.	$-\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}$	$=$	$-\hat{\boldsymbol{k}}$	$\hat{\boldsymbol{k}}^2 \leadsto -1$
3.	$\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}$	$=$	$\hat{\boldsymbol{k}}$	cancel the -1

Exercise: Multiplication of Quaternion Units

$\hat{\imath}\hat{\jmath}$	$\hat{\jmath}\hat{k}$	$\hat{\imath}\hat{k}$	$\hat{\jmath}\hat{\imath}$	$\hat{k}\hat{\jmath}$	$\hat{k}\hat{\imath}$
$\hat{\imath}\hat{\jmath}\hat{k} = -1$ $\hat{\imath}\hat{\jmath}\hat{k}^2 = -\hat{k}$ $-\hat{\imath}\hat{\jmath} = -\hat{k}$ $\hat{\imath}\hat{\jmath} = \hat{k}$					

Multiplication of Quaternion Units

Summary

$*$	$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$
$\hat{\imath}$	$\hat{\imath}^2 = -1$	$\hat{\imath}\hat{\jmath} = \hat{k}$	$\hat{\imath}\hat{k} = -\hat{\jmath}$
$\hat{\jmath}$	$\hat{\jmath}\hat{\imath} = -\hat{k}$	$\hat{\jmath}^2 = -1$	$\hat{\jmath}\hat{k} = \hat{\imath}$
$\hat{k}$	$\hat{k}\hat{\imath} = \hat{\jmath}$	$\hat{k}\hat{\jmath} = -\hat{\imath}$	$\hat{k}^2 = -1$

Example: Complex Multiplication

1. $(a_w + a_x \hat{\imath})(b_w + b_x \hat{\imath})$
2. $(a_w + a_x \hat{\imath}) b_w + (a_w + a_x \hat{\imath}) b_x \hat{\imath}$
3. $a_w b_w + a_x b_w \hat{\imath} + a_w b_x \hat{\imath} + a_x b_x \hat{\imath}^2$
4. $(a_w b_w - a_x b_x) + (a_x b_w + a_w b_x) \hat{\imath}$

Exercise: Quaternion Multiplication

1. $(a_w + a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}}) \otimes (b_w + b_x\hat{\boldsymbol{\imath}} + b_y\hat{\boldsymbol{\jmath}} + b_z\hat{\boldsymbol{k}})$

Complex Multiplication Matrix

Multiplication:

$$(a_w + a_x \hat{\imath}) \otimes (b_w + b_x \hat{\imath}) = a_w b_w - a_x b_x + (a_x b_w + a_w b_x)\hat{\imath}$$

Multiplication Matrix:

$$\begin{bmatrix} a_w & -a_x \\ a_x & a_w \end{bmatrix} \begin{bmatrix} b_w \\ b_x \end{bmatrix} = \begin{bmatrix} a_w b_w - a_x b_x \\ a_x b_w + a_w b_x \end{bmatrix}$$

Quaternion Multiplication Matrix

Matrix Form:

$$x\hat{\imath} + y\hat{\jmath} + z\hat{\mathbf{k}} + w = \begin{bmatrix} x & y & z & w \end{bmatrix}^T$$

Multiplication:

$$a \otimes b = \begin{array}{l} (a_w b_x - a_z b_y + a_y b_z + a_x b_w)\hat{\imath} \\ +(a_z b_x + a_w b_y - a_x b_z + a_y b_w)\hat{\jmath} \\ +(-a_y b_x + a_x b_y + a_w b_z + a_z b_w)\hat{\mathbf{k}} \\ +(-a_x b_x - a_y b_y - a_z b_z + a_w b_w) \end{array}$$

Multiplication Matrix:

$$\begin{bmatrix} a_w & -a_z & a_y & a_x \\ a_z & a_w & -a_x & a_y \\ -a_y & a_x & a_w & a_z \\ -a_x & -a_y & -a_w & a_w \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \\ b_w \end{bmatrix} = \begin{bmatrix} a_w b_x - a_z b_y + a_y b_z + a_x b_w) \\ a_z b_x + a_w b_y - a_x b_z + a_y b_w \\ -a_y b_x + a_x b_y + a_w b_z + a_z b_w \\ -a_x b_x - a_y b_y - a_z b_z + a_w b_w \end{bmatrix}$$

Use: quaternions within a larger system of linear equations

Exercise: Quaternion Multiplication Commutativity

Conjugate

Complex:

$$
\begin{aligned}
c &= a + b\hat{\imath} \\
c^* &= a - b\hat{\imath} \\
cc^* &= a^2 + b^2
\end{aligned}
$$

Quaternion:

$$
\begin{aligned}
\hbar &= x\hat{\imath} + y\hat{\jmath} + z\hat{\boldsymbol{k}} + w \\
\hbar^* &= -x\hat{\imath} - y\hat{\jmath} - z\hat{\boldsymbol{k}} + w
\end{aligned}
$$

Conjugate

Multiplication

1. $\hbar\hbar^* = \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k} + w\right) \otimes \left(-x\hat{\imath} - y\hat{\jmath} - z\hat{k} + w\right)$

2. $$= \begin{array}{c} (-wx + xw - yz + zy)\hat{\imath} \\ +(-wy + xz + yw - zx)\hat{\jmath} \\ +(-wz - xy + yx + zw)\hat{k} \\ +(ww + xx + yy + zz) \end{array}$$

3. $= x^2 + y^2 + z^2 + w^2$

Cancel the imaginary part

Quaternion Norm and Inverse

Norm

$$|q| = \sqrt{q_x^2 + q_y^2 + q_z^2 + q_w^2}$$

Inverse

$$q^{-1} = \frac{q^*}{|q|^2} = \frac{-q_x\hat{\boldsymbol{\imath}} + -q_y\hat{\boldsymbol{\jmath}} + -q_z\hat{\boldsymbol{k}} + q_w}{q_x^2 + q_y^2 + q_z^2 + q_w^2}$$

$$q \otimes q^{-1} = 1$$

When $|q| = 1$, then $q^* = q^{-1}$ and $q \otimes q^* = 1$

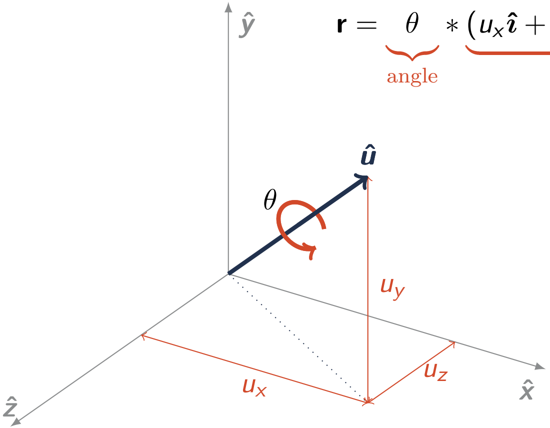
Quaternion Algebra

$$p = p_w + p_x\hat{\boldsymbol{\imath}} + p_y\hat{\boldsymbol{\jmath}} + p_z\hat{\boldsymbol{k}}$$

$$q = q_w + q_x\hat{\boldsymbol{\imath}} + q_y\hat{\boldsymbol{\jmath}} + q_z\hat{\boldsymbol{k}}$$

Axis-Angle and Rotation Vector

$$\mathbf{r} = \underbrace{\theta}_{\text{angle}} * \underbrace{\left(u_x\hat{\boldsymbol{\imath}} + u_y\hat{\boldsymbol{\jmath}} + u_z\hat{\boldsymbol{k}}\right)}_{\text{axis}}$$



Any 3D rotation: an angle θ about an axis $\hat{\boldsymbol{u}}$

Quaternion Rotations

Complex:

$$(x_1 + y_1\hat{\boldsymbol{\imath}}) = e^{\theta\hat{\boldsymbol{\imath}}} \overbrace{(x_0 + y_0\hat{\boldsymbol{\imath}})}^{\text{2D point}}$$

Quaternion:

$$\begin{aligned}(x_1\hat{\boldsymbol{\imath}} + y_1\hat{\boldsymbol{\jmath}} + z_1\hat{\boldsymbol{k}}) &= \exp\left(\frac{\theta}{2}\hat{\boldsymbol{u}}\right) \overbrace{(x_0\hat{\boldsymbol{\imath}} + y_0\hat{\boldsymbol{\jmath}} + z_0\hat{\boldsymbol{k}})}^{\text{3D point}} \exp\left(-\frac{\theta}{2}\hat{\boldsymbol{u}}\right) \\ &= \exp\left(\frac{\theta}{2}\hat{\boldsymbol{u}}\right) (x_0\hat{\boldsymbol{\imath}} + y_0\hat{\boldsymbol{\jmath}} + z_0\hat{\boldsymbol{k}}) \left(\exp\left(\frac{\theta}{2}\hat{\boldsymbol{u}}\right)\right)^*\end{aligned}$$

- Pre/Post multiply to keep point in vector/imaginary part
- Each contributes half the angle

Quaternion Exponential

Complex:

$$e^{\theta \hat{\imath}} = \hat{\imath} \sin\theta + \cos\theta$$

Pure Quaternion:

$$\exp\left(\frac{\mathbf{r}}{2}\right) = \exp\left(\frac{\theta}{2}\hat{\boldsymbol{u}}\right) = \sin\frac{\theta}{2}\left(u_x\hat{\imath} + u_y\hat{\jmath} + u_z\hat{\boldsymbol{k}}\right) + \cos\frac{\theta}{2}$$

General Quaternion:

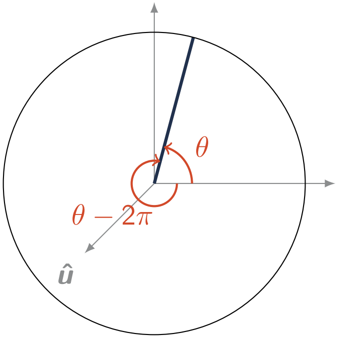
$$\exp x\hat{\imath} + y\hat{\jmath} + z\hat{\boldsymbol{k}} + w = \exp\left(\vec{v} + w\right) = e^{w}\left(\frac{\sin|v|}{|v|}\vec{v} + \cos|v|\right)$$

Exercise: Axis-Angle to Quaternion

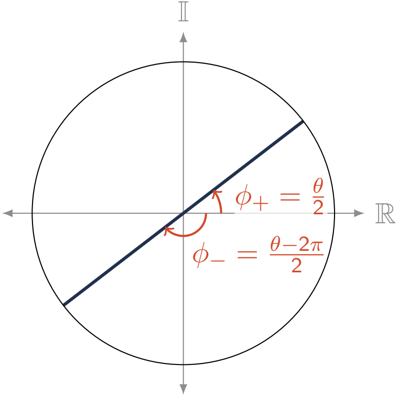
1. $\theta = \pi$ and $\hat{\boldsymbol{u}} = \hat{\boldsymbol{\imath}}$
   $\exp\left(\frac{\pi}{2}\hat{\boldsymbol{\imath}}\right) \quad \rightsquigarrow \quad \sin\frac{\pi}{2}\hat{\boldsymbol{\imath}} + \cos\frac{\pi}{2} \quad \rightsquigarrow \quad \hat{\boldsymbol{\imath}}$
2. $\theta = \frac{\pi}{2}$ and $\hat{\boldsymbol{u}} = \hat{\boldsymbol{k}}$
3. $\theta = -\frac{3\pi}{2}$ and $\hat{\boldsymbol{u}} = \hat{\boldsymbol{k}}$
4. $\theta = 0$ and $\hat{\boldsymbol{u}} = \hat{\boldsymbol{\imath}}$

Double Cover

Spatial



Quaternion



Quaternion Logarithm

Complex: $\ln(a + b\hat{\imath}) = \ln\left(\sqrt{a^2 + b^2}\right) + \hat{\imath}\,\mathrm{atan2}(b, a)$

Quaternion: $h = x\hat{\imath} + y\hat{\jmath} + z\hat{k} + w + \vec{v} + w$

$$\phi = \mathrm{atan2}(|v|, w)$$

$$\ln h = \frac{\phi}{|v|}\vec{v} + \ln|h|$$

$$\frac{\phi}{|v|} = \frac{\left(\frac{\phi}{|h|}\right)}{\left(\frac{|v|}{|h|}\right)} = \frac{\left(\frac{\phi}{|h|}\right)}{\sin\phi} = \frac{\frac{\phi}{\sin\phi}}{|h|}$$

$$\frac{\sin\phi}{\phi} = 1 - \frac{\phi^2}{6} + \frac{\phi^4}{120} - \frac{\phi^6}{5040} + \frac{\phi^8}{362880} - \frac{\phi^{10}}{39916800} + \ldots$$

Velocities and Derivatives

Angular Velocity: $\boldsymbol{\omega} = \omega_x \hat{\boldsymbol{\imath}} + \omega_y \hat{\boldsymbol{\jmath}} + \omega_z \hat{\boldsymbol{k}}$

Quaternion Derivative:

- $\dot{h} = \omega_x \hat{\boldsymbol{\imath}} + \omega_y \hat{\boldsymbol{\jmath}} + \omega_z \hat{\boldsymbol{k}}$
  $= \frac{1}{2} \boldsymbol{\omega} \otimes h$
- $\boldsymbol{\omega} = 2 \, \dot{h} \otimes h^*$

Product Rule: $\frac{d}{dt} \left(a(t) \otimes b(t) \right) = \left(\dot{a}(t) \otimes b(t) \right) + \left(a(t) \otimes \dot{b}(t) \right)$

Acceleration:

- $\ddot{h} = \frac{1}{2} \left(\dot{\boldsymbol{\omega}} \otimes h + \boldsymbol{\omega} \otimes \dot{h} \right)$
- $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = 2 \left(\ddot{h} \otimes h^* + \dot{h} \otimes \dot{h}^* \right)$

Integration

Quaternions as Linear ODE

$$\frac{d}{dt}\hbar = \frac{1}{2}\boldsymbol{\omega} \otimes \hbar$$

$$\hbar_1 = \exp\left(\frac{\boldsymbol{\omega}\Delta t}{2}\right) \otimes \hbar_0$$

Proportional Control

Given:
- Actual orientation $\hbar_a$
- Desired orientation $\hbar_d$

Find: Velocity command ω_u to reach $\hbar_d$

Solution:

1. $\hbar_d = \exp\left(\frac{\omega_u \Delta t}{2}\right) \otimes \hbar_a$
2. $\hbar_d \otimes (\hbar_a)^* = \exp\left(\frac{\omega_u \Delta t}{2}\right)$
3. $\ln\left(\hbar_d \otimes (\hbar_a)^*\right) = \frac{\omega_u \Delta t}{2}$
4. $\frac{2\ln(\hbar_d \otimes (\hbar_a)^*)}{\Delta t} = \omega_u$
5. $\rightsquigarrow \boxed{\omega_u \propto \ln\left(\hbar_d \otimes (\hbar_a)^*\right)}$

Note 0: William Rowan Hamilton

1805-1865

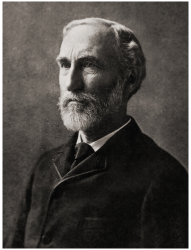
Brougham (Broom) Bridge, Dublin

Note 1: Gibb's Vector Analysis

$$a \otimes b = \begin{matrix} (a_y b_z - a_z b_y + a_w b_x + b_w a_x)\hat{\imath} + \\ (a_z b_x - a_x b_z + a_w b_y + b_w a_y)\hat{\jmath} + \\ (a_x b_y - a_y b_x + a_w b_z + b_w a_z)\hat{k} + \\ (a_w b_w - a_x b_x - a_y b_y - a_z b_z) \end{matrix} = \begin{pmatrix} a_v \times b_v + a_w b_v + b_w a_v \\ a_w b_w - a_v \cdot b_v \end{pmatrix}$$

$$\left(a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}\right) \times \left(b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}\right) = \begin{matrix} (a_y b_z - a_z b_y)\ \hat{\imath} \\ + (a_z b_x - a_x b_z)\ \hat{\jmath} \\ + (a_x b_y - a_y b_x)\ \hat{k} \end{matrix}$$

$$\left(a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}\right) \cdot \left(b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}\right) = a_x b_x + a_y b_y + a_z b_z$$



J. Willard Gibbs
1839-1903

Counterpoint:

Quaternions offer **computational** *advantages for rotations.*

Summary

References

- Lynch & Park. Modern Robotics. 2017. Ch. 3.2, Appendix B.
- James Diebel. Representing attitude: Euler angles, unit quaternions, and rotation vectors. Technical report, Stanford University, 2006.