Euclidean Transformation (Pre Lecture)

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Introduction

Transformations

- Each robot joint produces a 3D transformation (displacement)
- 3D transformation has rotation and (linear) translation
- Need to represent and compose (chain) transformations

Outcomes

- Visualize transformations and chaining
- Apply **dual quaternions** to represent transformations
- Contrast dual quaternions and other representations for transformation
- Construct transformations for a robot arm

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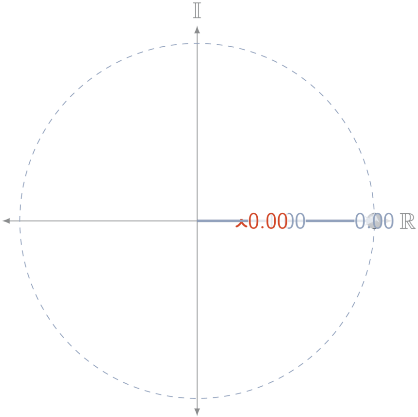
Outline

Planar Rotation

Complex Numbers

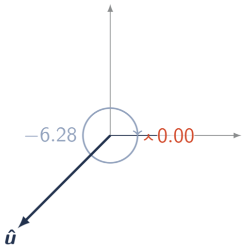


$$\exp(\;0.00\;\hat{\imath}) = 1.00 + 0.00\;\hat{\imath}$$

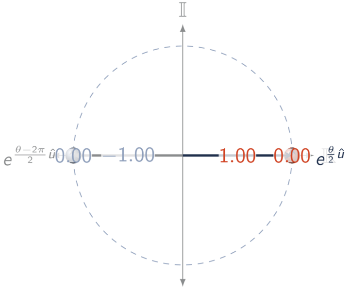
3D Rotation

Quaternions

Axis-Angle



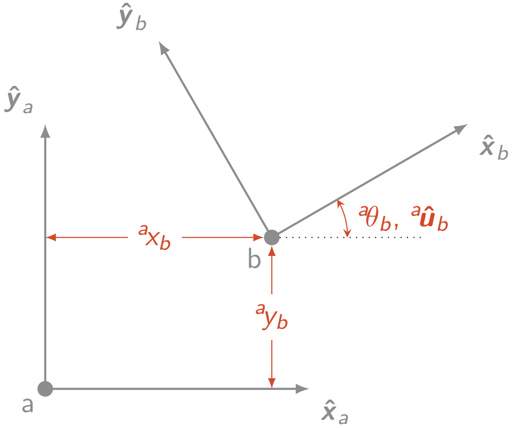
Quaternion



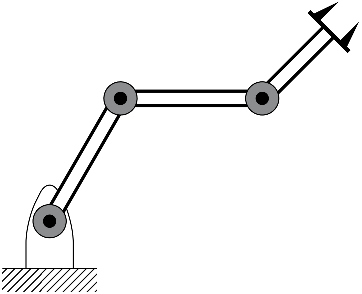
$$\begin{aligned} \theta &= 0.00 \\ \theta - 2\pi &= -6.28 \end{aligned}$$

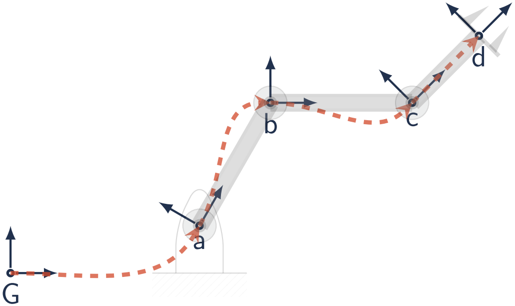
$$\begin{aligned} & 0.00\ \hat{\boldsymbol{u}} + 1.00 \\ \sim\ & 0.00\ \hat{\boldsymbol{u}} + -1.00 \end{aligned}$$

Local Coordinate Frames

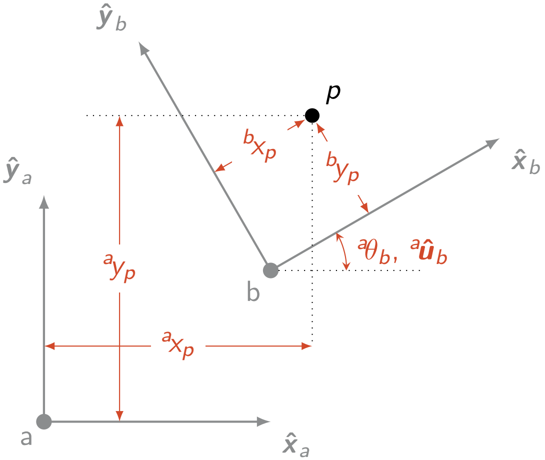


Robots are Local Frames





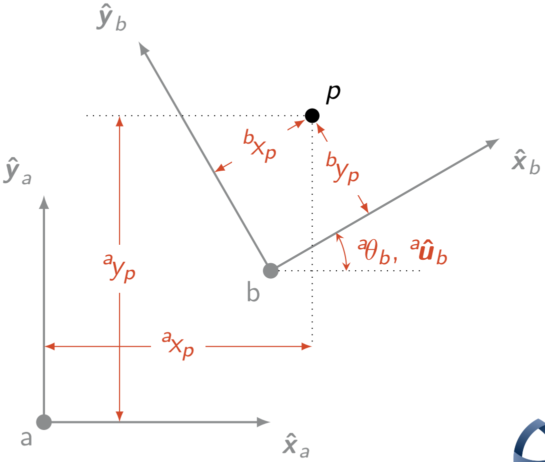
Transformations



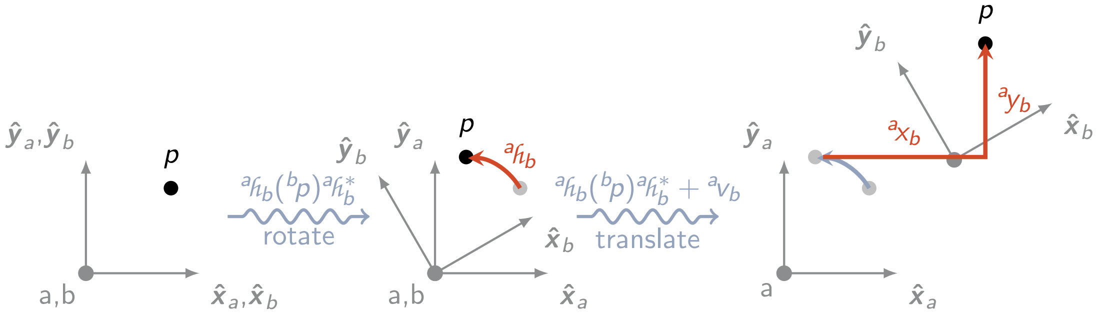
- p in coordinate frame a:
  $p = ({}^a x_p, {}^a y_p)$
- p in coordinate frame b:
  $p = ({}^b x_p, {}^b y_p)$

A notation convention

- ${}^{\text{parent}}X_{\text{child}}$
- ${}^{a}\mathcal{h}_b$: rotation quaternion from frame a to b
- ${}^{a}v_b$: translation vector from frame a to b



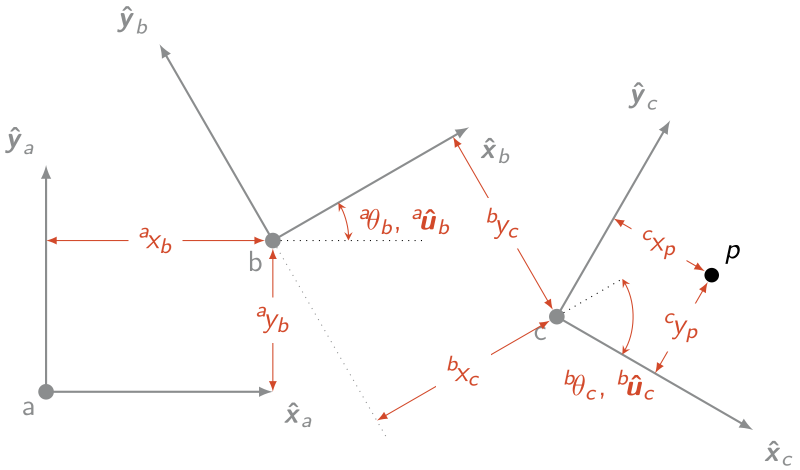
Transforming a Point



$$ {}^a p = \underbrace{({}^a h_b) \otimes ({}^b p) \otimes ({}^a h_b)^*}_{\text{rotation}} + \underbrace{{}^a v_b}_{\text{translation}} $$

Chaining Transforms

Geometric Illustration



Chaining Transforms

Algebraic Solution

- Transform ${}^{c}p$ to ${}^{b}p$: $\quad {}^{b}p = \left({}^{b}\hbar_{c}\right) \otimes \left({}^{c}p\right) \otimes \left({}^{b}\hbar_{c}\right)^{*} + {}^{b}v_{c}$
- Transform ${}^{b}p$ to ${}^{a}p$: $\quad {}^{a}p = \left({}^{a}\hbar_{b}\right) \otimes \left({}^{b}p\right) \otimes \left({}^{a}\hbar_{b}\right)^{*} + {}^{a}v_{b}$
- Transform ${}^{b}p$ to ${}^{a}p$:

1. $${}^{a}p = \left({}^{a}\hbar_{b}\right) \otimes \overbrace{\left(\left({}^{b}\hbar_{c}\right) \otimes \left({}^{c}p\right) \otimes \left({}^{b}\hbar_{c}\right)^{*} + {}^{b}v_{c}\right)}^{{}^{b}p} \otimes \left({}^{a}\hbar_{b}\right)^{*} + {}^{a}v_{b}$$
2. $${}^{a}p = \left(\left({}^{a}\hbar_{b}\right) \otimes \left({}^{b}\hbar_{c}\right) \otimes \left({}^{c}p\right) \otimes \left({}^{b}\hbar_{c}\right)^{*} + \left({}^{a}\hbar_{b}\right) \otimes {}^{b}v_{c}\right) \otimes \left({}^{a}\hbar_{b}\right)^{*} + {}^{a}v_{b}$$
3. $${}^{a}p = \left({}^{a}\hbar_{b}\right) \otimes \left({}^{b}\hbar_{c}\right) \otimes \left({}^{c}p\right) \otimes \left({}^{b}\hbar_{c}\right)^{*} \otimes \left({}^{a}\hbar_{b}\right)^{*} + \left({}^{a}\hbar_{b}\right) \otimes {}^{b}v_{c} \otimes \left({}^{a}\hbar_{b}\right)^{*} + {}^{a}v_{b}$$
4. $${}^{a}p = \underbrace{\left({}^{a}\hbar_{b}\right) \otimes \left({}^{b}\hbar_{c}\right)}_{{}^{a}\hbar_{c}} \otimes \left({}^{c}p\right) \otimes \underbrace{\left({}^{a}\hbar_{b} \otimes {}^{b}\hbar_{c}\right)^{*}}_{{}^{a}\hbar_{c}} + \underbrace{\left({}^{a}\hbar_{b}\right) \otimes {}^{b}v_{c} \otimes \left({}^{a}\hbar_{b}\right)^{*} + {}^{a}v_{b}}_{{}^{a}v_{c}}$$

- ${}^{a}\hbar_{c} = \left({}^{a}\hbar_{b} \otimes {}^{b}\hbar_{c}\right)$ and ${}^{a}v_{c} = \left({}^{a}\hbar_{b}\right) \otimes {}^{b}v_{c} \otimes \left({}^{a}\hbar_{b}\right)^{*} + {}^{a}v_{b}$

Outline

Dual Axiom

$$\varepsilon^2 = 0 \qquad \wedge \qquad \varepsilon \neq 0$$

Example:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Dual Numbers

$$\varepsilon^2 = 0 \qquad \wedge \qquad \varepsilon \neq 0$$

$$\tilde{n} = \overbrace{\underbrace{n_r}_{\text{real}} + \underbrace{n_d \varepsilon}_{\text{dual}}}^{\text{dual number}}$$

Dual Number Multiplication

0.	$\tilde{a} \otimes \tilde{b}$	$=$	$(a_r + a_d\varepsilon) \otimes (b_r + b_d\varepsilon)$	Multiplication Expression
1.		$=$	$(a_r + a_d\varepsilon)b_r + (a_r + a_d\varepsilon)b_d\varepsilon$	Distribute $\tilde{a}$ over $\tilde{b}$
2.		$=$	$a_r b_r + a_d b_r\varepsilon + a_r b_d\varepsilon + a_d b_d\varepsilon^2$	Distribute a_r and a_d over $\tilde{b}$
3.		$=$	$a_r b_r + a_d b_r\varepsilon + a_r b_d\varepsilon + \cancel{a_d b_d\varepsilon^2} \rightarrow 0$	Cancel $\varepsilon^2 = 0$
4.		$=$	$a_r b_r + (a_r b_d + a_d b_r)\varepsilon$	Simplify

Dual Conjugate

- $(r + d\varepsilon)^{\bullet} = r - d\varepsilon$
- Multiplication by conjugate:
  1. $(r + d\varepsilon)(r - d\varepsilon)$
  2. $= r^2 + rd\varepsilon - rd\varepsilon$
  3. $= r^2$

Cancels the dual part

Dual Number Taylor Series

Taylor Series: $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$

Dual Number Taylor Series: Evaluate Taylor series at the real part:

0.	$f(a+b\varepsilon)$	$=$	$f(a) + \frac{f'(a)}{1!}(b\varepsilon) + \frac{f''(a)}{2!}(b\varepsilon)^2 + \frac{f'''(a)}{3!}b\varepsilon)^3 + \dots$	Taylor Series
1.	$f(a+b\varepsilon)$	$=$	$f(a) + \frac{f'(a)}{1!}(b\varepsilon) + \cancelto{0}{\frac{f''(a)}{2!}(b\varepsilon)^2} + \cancelto{0}{\frac{f'''(a)}{3!}b\varepsilon)^3} + \cancelto{0}{\dots}$	$\varepsilon^2 = 0$
2.	$f(a+b\epsilon)$	$=$	$f(a) + bf'(a)\varepsilon$	Result

Higher-order dual terms cancel

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Example: Dual Number Transcendental Functions

- **Taylor Series:** $f(a + b\epsilon) = f(a) + bf'(a)\varepsilon$
- **Exponential:** $e^{r+d\varepsilon} \quad \rightsquigarrow \quad e^r + d\left(e^r\right)' \varepsilon \quad \rightsquigarrow \quad e^r + de^r \varepsilon$

Exercise: Dual Number Transcendental Functions

Taylor Series: $f(a+b\epsilon) = f(a) + bf'(a)\varepsilon$

Exponential: $e^{r+d\varepsilon} \leadsto e^r + d\left(e^r\right)'\varepsilon \leadsto e^r + de^r\varepsilon$

Logarithm: $\ln\left(r+d\varepsilon\right) \leadsto$

Sine: $\sin\left(r+d\varepsilon\right) \leadsto$

Cosine: $\cos\left(r+d\varepsilon\right) \leadsto$

Square Root: $\sqrt{r+d\varepsilon} \leadsto$

Dual Quaternions

Dual Number

$$r + d\varepsilon$$

Quaternion

$$x\hat{\imath} + y\hat{\jmath} + z\hat{k} + w$$

$$\left(r_x\hat{\imath} + r_y\hat{\jmath} + r_z\hat{k} + r_w\right) + \left(d_x\hat{\imath} + d_y\hat{\jmath} + d_z\hat{k} + d_w\right)\varepsilon$$
$$= (r_x + d_x\varepsilon)\,\hat{\imath} + (r_y + d_y\varepsilon)\,\hat{\jmath} + (r_z + d_z\varepsilon)\,\hat{k} + (r_w + d_w\varepsilon)$$

Dual Quaternion

8 factors for the combinations of real, quaternion, and dual parts.

Terminology Redux

Ordinary Quaternion:

$$\hbar = \underbrace{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}}_{\text{vector}} + \underbrace{w}_{\text{scalar}}$$

Dual Quaternion:

$$\mathcal{S} = \underbrace{\left(r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}} + r_z\hat{\boldsymbol{k}} + r_w\right)}_{\text{real part}} + \underbrace{\left(d_x\hat{\boldsymbol{\imath}} + d_y\hat{\boldsymbol{\jmath}} + d_z\hat{\boldsymbol{k}} + d_w\right)\varepsilon}_{\text{dual part}}$$

Dual Quaternion Multiplication

- $a = (a_r + a_d\varepsilon) = \left(\left(a_{rx}\hat{\imath} + a_{ry}\hat{\jmath} + a_{rz}\hat{k} + a_{rw}\right) + \left(a_{dx}\hat{\imath} + a_{dy}\hat{\jmath} + a_{dz}\hat{k} + a_{dw}\right)\varepsilon\right)$
- $b = (b_r + b_d\varepsilon) = \left(\left(b_{rx}\hat{\imath} + b_{ry}\hat{\jmath} + b_{rz}\hat{k} + b_{rw}\right) + \left(b_{dx}\hat{\imath} + b_{dy}\hat{\jmath} + b_{dz}\hat{k} + b_{dw}\right)\varepsilon\right)$
- Multiplication:

0.	$a \otimes b$	$=$	$(a_r + a_d\varepsilon) \otimes (b_r + b_d\varepsilon)$	Multiplication Expression
1.		$=$	$(a_r + a_d\varepsilon) \otimes b_r + (a_r + a_d\varepsilon) \otimes b_d\varepsilon$	Distribute a over b
2.		$=$	$a_r \otimes b_r + a_d \otimes b_r\varepsilon + a_r \otimes b_d\varepsilon + a_d \otimes b_d\varepsilon^2$	Distribute a_r and a_d over b
3.		$=$	$a_r \otimes b_r + a_d \otimes b_r\varepsilon + a_r \otimes b_d\varepsilon + \cancel{a_d b_d\varepsilon^2} \rightarrow 0$	Cancel $\varepsilon^2 = 0$
4.		$=$	$a_r \otimes b_r + (a_r \otimes b_d + a_d \otimes b_r)\varepsilon$	Simplify

Dual number multiplication, but with quaternion multiplies

Dual Quaternion Conjugates

Quaternion Conjugate:

$$(h + d\varepsilon)^* = h^* + d^*\varepsilon$$

Dual Conjugate:

$$(h + d\varepsilon)^\bullet = h - d\varepsilon$$

Joint Conjugate:

$$(h + d\varepsilon)^\diamond = ((h + d\varepsilon)^*)^\bullet = h^* - d^*\varepsilon$$

Dual Quaternions Transformations

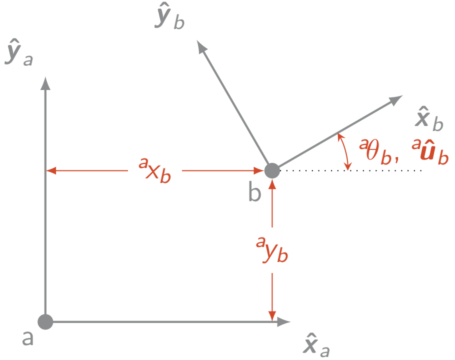
Illustration

Rotation: ${}^{a}h_{b} = \exp\left(\frac{1}{2}\theta\hat{\boldsymbol{u}}\right)$

Translation: ${}^{a}v_{b} = {}^{a}x_{b}\hat{\boldsymbol{\imath}} + {}^{a}y_{b}\hat{\boldsymbol{\jmath}} + {}^{a}z_{b}\hat{\boldsymbol{k}}$

Transform: ${}^{a}S_{b} = \left({}^{a}h_{b}\right) + \left(\frac{1}{2}{}^{a}v_{b} \otimes {}^{a}h_{b}\right)\varepsilon$

- $d = \frac{1}{2}v \otimes h$
- $v = 2d \otimes h^{*}$



Dual Quaternions Transformations

Algebra

$$\text{Rotation:} \quad {}^{a}p = {}^{a}h_b \otimes \overbrace{\left(p_x\hat{\imath} + p_y\hat{\jmath} + p_z\hat{\boldsymbol{k}}\right)}^{\text{point}} \otimes \left({}^{a}h_b\right)^*$$

$$\text{Transform:} \quad {}^{a}p = {}^{a}S_b \otimes \overbrace{\left(1 + \left(p_x\hat{\imath} + p_y\hat{\jmath} + p_z\hat{\boldsymbol{k}}\right)\varepsilon\right)}^{\text{point}} \otimes \left({}^{a}S_b\right)^\diamond$$

1. $= \left(h + d\varepsilon\right)\left(1 + p\varepsilon\right)\left(h + d\varepsilon\right)^\diamond$
2. $= \left(h + \left(d + hp\right)\varepsilon\right)\left(h^* - d^*\varepsilon\right)$
3. $= hh^* + \left(\left(d + hp\right)h^* - hd^*\right)\varepsilon$
4. $= 1 + (\underbrace{hph^*}_{\text{rotate}} + \underbrace{dh^* - hd^*}_{\text{translate}})\varepsilon$

Transformation Formula

Simplified

Point: $^{b}p = p_x\hat{\boldsymbol{\imath}} + p_y\hat{\boldsymbol{\jmath}} + p_z\hat{\boldsymbol{k}}$

Transform: $^{a}\mathcal{S}_b = \mathcal{h} + \mathcal{d}\varepsilon$

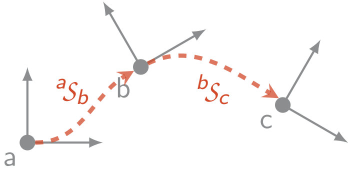
Result:

$$^{a}p = {}^{a}\mathcal{S}_b \otimes \left(1 + {}^{b}p\varepsilon\right) \otimes \left({}^{a}\mathcal{S}_b\right)^{\diamond}$$

$$= \left(\mathcal{h} \otimes {}^{b}p + 2\mathcal{d}\right) \otimes \mathcal{h}^{*}$$

Dual Quaternion Chaining

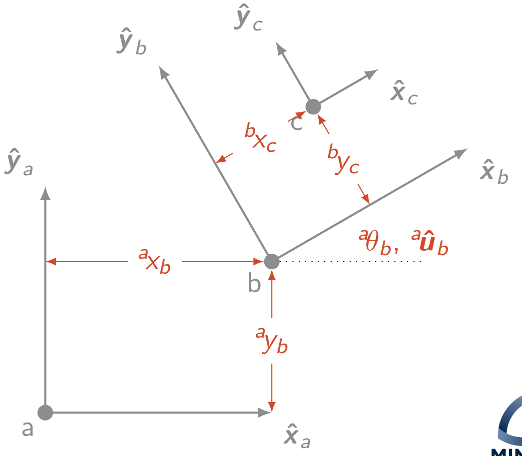
- $$\begin{aligned} {}^a\mathcal{S}_c &= \left({}^a\mathcal{S}_b \otimes {}^b\mathcal{S}_c \right) = \left(\left({}^a\mathcal{h}_b + {}^a\mathcal{d}_b \varepsilon \right) \otimes \left({}^b\mathcal{h}_c + {}^b\mathcal{d}_c \varepsilon \right) \right) \\ &= \left(\left({}^a\mathcal{h}_b \otimes {}^b\mathcal{h}_c \right) + \left({}^a\mathcal{h}_b \otimes {}^b\mathcal{d}_c + {}^a\mathcal{d}_b \otimes {}^b\mathcal{h}_c \right) \varepsilon \right) \end{aligned}$$
- Transform Multiply:
  1. $${}^a\mathcal{S}_c = \left({}^a\mathcal{h}_b + \tfrac{1}{2}\, {}^a\mathcal{v}_b \, {}^a\mathcal{h}_b \varepsilon \right) \otimes \left({}^b\mathcal{h}_c + \tfrac{1}{2}\, {}^b\mathcal{v}_c \, {}^b\mathcal{h}_c \varepsilon \right)$$
  2. $$= \underbrace{\left({}^a\mathcal{h}_b \, {}^b\mathcal{h}_c \right)}_{\text{rotation}} + \underbrace{\frac{1}{2} \left({}^a\mathcal{h}_b \, {}^b\mathcal{v}_c \, {}^b\mathcal{h}_c + {}^a\mathcal{v}_b \, {}^a\mathcal{h}_b \, {}^b\mathcal{h}_c \right) \varepsilon}_{\text{translation}}$$
- Extract Translation: $v = 2\mathcal{d} \otimes \mathcal{h}^*$
  1. $${}^a\mathcal{v}_c = 2 \left(\tfrac{1}{2} \left({}^a\mathcal{h}_b \, {}^b\mathcal{v}_c \, {}^b\mathcal{h}_c + {}^a\mathcal{v}_b \, {}^a\mathcal{h}_b \, {}^b\mathcal{h}_c \right) \right) \otimes \left({}^a\mathcal{h}_b \, {}^b\mathcal{h}_c \right)^*$$
  2. $${}^a\mathcal{v}_c = \left({}^a\mathcal{h}_b \, {}^b\mathcal{v}_c \, {}^b\mathcal{h}_c + {}^a\mathcal{v}_b \, {}^a\mathcal{h}_b \, {}^b\mathcal{h}_c \right) \otimes \left({}^b\mathcal{h}_c \right)^* \left({}^a\mathcal{h}_b \right)^*$$
  3. $${}^a\mathcal{v}_c = \left({}^a\mathcal{h}_b \, {}^b\mathcal{v}_c \, {}^b\mathcal{h}_c \left({}^b\mathcal{h}_c \right)^* \left({}^a\mathcal{h}_b \right)^* + {}^a\mathcal{v}_b \, {}^a\mathcal{h}_b \, {}^b\mathcal{h}_c \left({}^b\mathcal{h}_c \right)^* \left({}^a\mathcal{h}_b \right)^* \right)$$
  4. $${}^a\mathcal{v}_c = {}^a\mathcal{h}_b \otimes {}^b\mathcal{v}_c \otimes \left({}^a\mathcal{h}_b \right)^* + {}^a\mathcal{v}_b$$



Dual Quaternion Transformation as Chaining

Illustration

- ${}^{a}\mathcal{S}_b = \mathcal{h} + \mathcal{d}\varepsilon$
- ${}^{b}p = {}^{b}x_c\hat{\imath} + {}^{b}y_c\hat{\jmath} + {}^{b}z_c\hat{\boldsymbol{k}}$
- ${}^{b}\mathcal{S}_c = 1 + \frac{1}{2}{}^{b}p\varepsilon$
- Chain Transforms:
  1. ${}^{a}\mathcal{S}_c = {}^{a}\mathcal{S}_b \otimes {}^{b}\mathcal{S}_c$
  2. $= (\mathcal{h} + \mathcal{d}\varepsilon) \otimes \left(1 + \frac{1}{2}{}^{b}p\varepsilon\right)$
  3. $= \mathcal{h} + \left(\mathcal{d} + \frac{1}{2}\mathcal{h} \otimes {}^{b}p\right)\varepsilon$
- Extract Point: $v = 2\mathcal{d} \otimes \mathcal{h}^{*}$
  1. ${}^{a}v = 2\left(\mathcal{d} + \frac{1}{2}\mathcal{h} \otimes {}^{b}p\right) \otimes \mathcal{h}^{*}$
  2. $= \left(2\mathcal{d} + \mathcal{h} \otimes {}^{b}p\right) \otimes \mathcal{h}^{*}$



Derivation: Dual Quaternion Exponential (1/4)

Ordinary Quaternion: $\mathcal{h} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} + w$

$$\phi = \sqrt{x^2 + y^2 + z^2}$$

$$e^{\mathcal{h}} = e^{w}\left(\frac{\sin\phi}{\phi}\left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}\right) + \cos\phi\right)$$

Dual Quaternion: $\mathcal{S} = \left(\mathcal{h}_x\hat{\boldsymbol{\imath}} + \mathcal{h}_y\hat{\boldsymbol{\jmath}} + \mathcal{h}_y\hat{\boldsymbol{k}} + \mathcal{h}_w\right) + \left(\mathcal{d}_x\hat{\boldsymbol{\imath}} + \mathcal{d}_y\hat{\boldsymbol{\jmath}} + \mathcal{d}_z\hat{\boldsymbol{k}} + \mathcal{d}_w\right)\varepsilon$

$$\tilde{\phi} = \sqrt{(\mathcal{h}_x + \mathcal{d}_x\epsilon)^2 + (\mathcal{h}_y + \mathcal{d}_z\epsilon)^2 + (\mathcal{h}_y + \mathcal{d}_z\epsilon)^2}$$

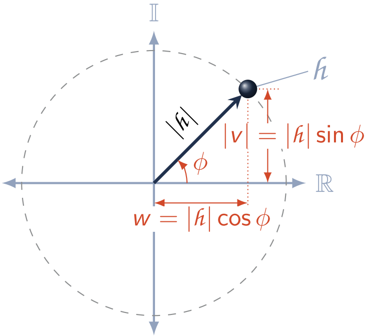
$$e^{\mathcal{S}} = e^{\mathcal{h}_w + \mathcal{d}_w\varepsilon}\left(\frac{\sin\tilde{\phi}}{\tilde{\phi}}\left((\mathcal{h}_x + \mathcal{d}_x\varepsilon)\,\hat{\boldsymbol{\imath}} + (\mathcal{h}_y + \mathcal{d}_y\varepsilon)\,\hat{\boldsymbol{\jmath}} + (\mathcal{h}_y + \mathcal{d}_z\varepsilon)\,\hat{\boldsymbol{k}}\right) + \cos\tilde{\phi}\right)$$

Derivation: Dual Quaternion Exponential (2/4)

Dual Angle $\tilde{\phi}$

$$\hbar = \vec{v} + w$$

- $\tilde{\phi} = \sqrt{(\hbar_x + d_x\epsilon)^2 + (\hbar_y + d_z\epsilon)^2 + (\hbar_y + d_z\epsilon)^2}$
  $= \sqrt{\left(\hbar_x^2 + \hbar_y^2 + \hbar_y^2\right) + 2\left(\hbar_x d_x + \hbar_y d_y + \hbar_y d_z\right)\varepsilon}$
  $= \sqrt{\hbar_x^2 + \hbar_y^2 + \hbar_y^2} + \frac{\hbar_x d_x + \hbar_y d_y + \hbar_y d_z}{\sqrt{\hbar_x^2+\hbar_y^2+\hbar_y^2}}\varepsilon$
  $= \phi + \frac{\gamma}{\phi}\varepsilon$
- $\cos\tilde{\phi} = \cos\phi - \frac{\gamma}{\phi}\sin(\phi)\varepsilon = c - \frac{\gamma}{\phi}s\varepsilon$
- $\sin\tilde{\phi} = \sin\phi + \frac{\gamma}{\phi}\cos(\phi)\varepsilon = s + \frac{\gamma}{\phi}c\varepsilon$



Derivation: Dual Quaternion Exponential (3/4)

Dual Sine Cardinal: $\left(\frac{\sin \tilde{\phi}}{\tilde{\phi}}\right)$

1. $\frac{\sin \tilde{\phi}}{\tilde{\phi}} = \frac{\sin(\phi)+\frac{\gamma}{\phi}\cos(\phi)\varepsilon}{\phi+\frac{\gamma}{\phi}\varepsilon}$
2. $= \left(\frac{\sin(\phi)+\frac{\gamma}{\phi}\cos(\phi)\varepsilon}{\phi+\frac{\gamma}{\phi}\varepsilon}\right)\left(\frac{\phi-\frac{\gamma}{\phi}\varepsilon}{\phi-\frac{\gamma}{\phi}\varepsilon}\right)$
3. $= \frac{\sin(\phi)\phi+\left(\phi\cos(\phi)\frac{\gamma}{\phi}-\sin(\phi)\frac{\gamma}{\phi}\right)\varepsilon}{\phi^2}$
4. $= \frac{\sin(\phi)}{\phi} + \gamma\left(\frac{\cos(\phi)-\frac{\sin(\phi)}{\phi}}{\phi^2}\right)\varepsilon$
5. $= \underbrace{\left(1-\frac{\phi^2}{6}+\frac{\phi^4}{120}+\dots\right)}_{(\sin\phi)/\phi} + \gamma\underbrace{\left(-\frac{1}{3}+\frac{\phi^2}{30}-\frac{\phi^4}{840}+\dots\right)}_{(\cos\phi-(\sin\phi)/\phi)/\phi^2}\varepsilon$

Derivation: Dual Quaternion Exponential (4/4)

Result

Dual Quaternion Exponential

$$e^{\mathcal{S}} = \left(e^{\hbar_w} + d_w e^{\hbar_w} \varepsilon\right) \left(\left(\frac{s}{\phi} \vec{\hbar}_v + c \right) + \left(\frac{s}{\phi} \vec{d}_v + \frac{c - \frac{s}{\phi}}{\phi^2} \gamma \vec{\hbar}_v - \frac{s}{\phi} \gamma \right) \varepsilon \right),$$

where

- $\gamma = \vec{\hbar}_v \cdot \vec{d}_v$.
- $\frac{\sin\phi}{\phi} = 1 - \frac{\phi^2}{6} + \frac{\phi^4}{120} + \ldots$
- $\frac{\cos(\phi) - \frac{\sin(\phi)}{\phi}}{\phi^2} = -\frac{1}{3} + \frac{\phi^2}{30} - \frac{\phi^4}{840} + \ldots$

Well-defined via Taylor series as $\phi \to 0$.

Dual Quaternion Logarithm

Quaternion: $\mathcal{h} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} + w = \vec{v} + w$

$$\ln \mathcal{h} = \frac{\phi}{|v|}\vec{v} + \ln|\mathcal{h}|, \quad \text{where } \phi = \operatorname{atan2}(|v|, w)$$

Dual Quaternion: $\mathcal{S} = \mathcal{h} + \mathcal{d}\varepsilon$

Dual Quaternion Logarithm

$$\ln \mathcal{S} = \frac{\phi}{|\mathcal{h}_v|}\vec{\mathcal{h}}_v + \ln|\mathcal{h}| + \left(\frac{\left(\vec{\mathcal{h}}_v \cdot \vec{\mathcal{d}}_v\right)\alpha - \mathcal{d}_w}{|\mathcal{h}|^2}\vec{\mathcal{h}}_v + \frac{\phi}{|\mathcal{h}_v|}\vec{\mathcal{d}}_v + \frac{\mathcal{h}\cdot\mathcal{d}}{|\mathcal{h}|^2} \right)\varepsilon,$$

$$\text{where} \quad \alpha = \frac{\mathcal{h}_w - \frac{\phi}{|\mathcal{h}_v|}|\mathcal{h}|^2}{|\mathcal{h}_v|^2} = \frac{\left(-\frac{2}{3} - \frac{\phi^2}{5} - \frac{\phi^4}{420} + \dots\right)}{|\mathcal{h}|}$$

Velocity and Derivatives

Quaternion Derivative: $\dot{h} = \frac{1}{2}\boldsymbol{\omega} \otimes h$

Dual Quaternion Derivative:

0.	$\mathcal{S}$	$=$	$h + \left(\frac{1}{2}v \otimes h\right)\varepsilon$	Dual Quat. Definition
1.	$\dot{\mathcal{S}}$	$=$	$\frac{d}{dt}\left(h + \left(\frac{1}{2}v \otimes h\right)\varepsilon\right)$	Time derivative
2.	$\dot{\mathcal{S}}$	$=$	$\dot{h} + \frac{d}{dt}\left(\frac{1}{2}v \otimes h\right)\varepsilon$	Addition Rule
3.	$\dot{\mathcal{S}}$	$=$	$\dot{h} + \frac{1}{2}\left(\dot{v} \otimes h + v \otimes \dot{h}\right)\varepsilon$	Product Rule
4.	$\dot{\mathcal{S}}$	$=$	$\frac{1}{2}\left(\omega \otimes h + \left(\dot{v} \otimes h + v \otimes \left(\frac{1}{2}\omega \otimes h\right)\right)\varepsilon\right)$	Substitute/Simplify

Dual Quaternion Product Rule

Ordinary Quaternion: $\frac{d}{dt}\left(a(t) \otimes b(t)\right) = \left(\dot{a}(t) \otimes b(t)\right) + \left(a(t) \otimes \dot{b}(t)\right)$

Dual Quaternion: ${}^{a}\mathcal{S}_{c} = {}^{a}\mathcal{S}_{b} \otimes {}^{b}\mathcal{S}_{c}$

1. $\frac{d}{dt}{}^{a}\mathcal{S}_{c} = \frac{d}{dt}\left({}^{a}\mathcal{S}_{b} \otimes {}^{b}\mathcal{S}_{c}\right)$
2. $\frac{d}{dt}{}^{a}\mathcal{S}_{c} = \frac{d}{dt}\left({}^{a}\mathcal{S}_{b}\right) \otimes {}^{b}\mathcal{S}_{c} + {}^{a}\mathcal{S}_{b} \otimes \frac{d}{dt}\left({}^{b}\mathcal{S}_{c}\right)$

Integration

Twist

- Factorization of the Dual Quaternion Derivative
- $\dot{\mathcal{S}} = \left(\frac{1}{2} \left(\omega \otimes h + \left(\dot{v} \otimes h + \frac{1}{2} v \otimes \omega \otimes h \right) \varepsilon \right) \right)$

  $\rightsquigarrow \quad \left(\frac{1}{2}\Omega \otimes \left(h + \left(\frac{1}{2}\vec{v} \otimes h \right) \varepsilon \right) \right)$

  1. $= \frac{1}{2} \left(\omega \otimes h + \left(\left(\dot{v} + \frac{1}{2} v \otimes \omega \right) \otimes h \right) \varepsilon \right)$
  2. $= \frac{1}{2} \left(\omega \otimes h + \left(\left(\dot{v} + \frac{1}{2} v \times \omega + \frac{1}{2} v \cdot \omega \right) \otimes h \right) \varepsilon \right)$
  3. $= \frac{1}{2} \left(\omega \otimes h + \left(\left(\dot{v} + \frac{1}{2}\omega \times v + \frac{1}{2}\omega \cdot v + v \times \omega \right) \otimes h \right) \varepsilon \right)$
  4. $= \frac{1}{2} \left(\omega \otimes h + \left(\left(\dot{v} + \frac{1}{2}\omega \otimes v + v \times \omega \right) \otimes h \right) \varepsilon \right)$
  5. $= \frac{1}{2} \left(\omega \otimes h + \left(\frac{1}{2}\omega \otimes v \otimes h + \left(\dot{v} + v \times \omega \right) \otimes h \right) \varepsilon \right)$
  6. $= \frac{1}{2} \left(\omega \otimes h + \left(\omega \otimes \left(\frac{1}{2} v \otimes h \right) + \left(\dot{v} + v \times \omega \right) \otimes h \right) \varepsilon \right)$
  7. $= \frac{1}{2} \left(\omega + \left(\dot{v} + v \times \omega \right) \varepsilon \right) \otimes \left(h + \frac{1}{2} v \otimes h \varepsilon \right)$
  8. $\dot{\mathcal{S}} = \frac{1}{2}\Omega \otimes \mathcal{S}$
- $\Omega = \omega + \left(\dot{v} + v \times \omega \right) \varepsilon$

Integration

- Dual Quaternions as Linear ODE
- $\frac{d}{dt}\mathcal{S} = \frac{1}{2}\,\Omega \otimes \mathcal{S}$
- $\mathcal{S}_1 = \exp\left(\frac{\Omega \Delta t}{2} \right) \otimes \mathcal{S}_0$

Outline

Rotation Matrices

Quaternion Multiplication:

$$p \otimes q = \overbrace{\begin{bmatrix} p_w & -p_z & p_y & p_x \\ p_z & p_w & -p_x & p_y \\ -p_y & p_x & p_w & p_z \\ -p_x & -p_y & -p_z & p_w \end{bmatrix}}^{\mathbf{P}_L} \begin{bmatrix} q_x \\ q_y \\ q_z \\ q_w \end{bmatrix} = \overbrace{\begin{bmatrix} q_w & q_z & -q_y & q_x \\ -q_z & q_w & q_x & q_y \\ q_y & -q_x & q_w & q_z \\ -q_x & -q_y & -q_z & q_w \end{bmatrix}}^{\mathbf{Q}_R} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}$$

Quaternion Rotation:

$$\hbar \otimes \vec{v} \otimes \hbar^* = (\mathbf{H}_L)(\vec{v} \otimes \hbar^*) = \overbrace{(\mathbf{H}_L)(\mathbf{H}^*{}_R)}^{\mathbf{R}} \begin{bmatrix} v_x & v_y & v_z & 0 \end{bmatrix}^T$$

$$\mathbf{R} = \begin{bmatrix} -h_z^2 - h_y^2 + h_x^2 + h_w^2 & 2h_x h_y - 2h_z h_w & 2h_x h_z + 2h_y h_w \\ 2h_z h_w + 2h_x h_y & -h_z^2 + h_y^2 - h_x^2 + h_w^2 & 2h_y h_z - 2h_x h_w \\ 2h_x h_z - 2h_y h_w & 2h_y h_z + 2h_x h_w & h_z^2 - h_y^2 - h_x^2 + h_w^2 \end{bmatrix}$$

Transformation Matrices

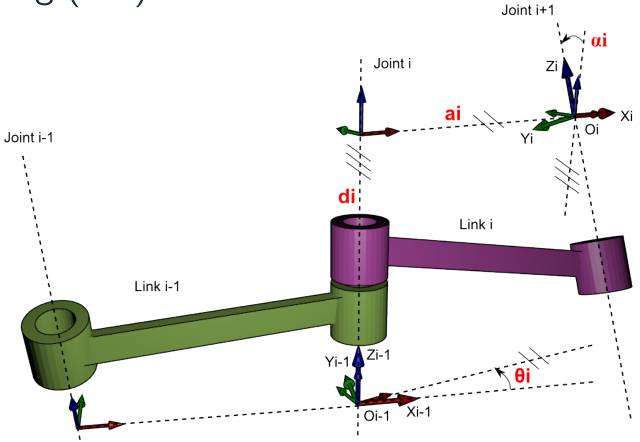
Transformation

- $^{a}p = {^{a}h_{b}} \otimes {^{b}p} \otimes ({^{a}h_{b}})^{*} + {^{a}v_{b}}$
- $= {^{a}\mathbf{R}_{b}} \begin{bmatrix} {^{b}x} \\ {^{y}x} \\ {^{z}x} \end{bmatrix} + \begin{bmatrix} ({^{a}v_{b}})_{x} \\ ({^{a}v_{b}})_{y} \\ ({^{a}v_{b}})_{z} \end{bmatrix}$
- $= \overbrace{\begin{bmatrix} {^{a}\mathbf{R}_{b}} & {^{a}\mathbf{v}_{b}} \\ 0 & 1 \end{bmatrix}}^{{^{a}\mathbf{T}_{b}}} \begin{bmatrix} {^{b}x} \\ {^{y}x} \\ {^{z}x} \\ 1 \end{bmatrix}$
- $^{a}\mathbf{p} = \left({^{a}\mathbf{T}_{b}}\right)\left({^{b}\mathbf{p}}\right)$

Chaining

- $^{a}\mathbf{p} = \overbrace{\left({^{a}\mathbf{T}_{b}}\right)\left({^{b}\mathbf{T}_{c}}\right)}^{{^{a}\mathbf{T}_{c}}}\left({^{c}\mathbf{p}}\right)$
- Chain:
  1. $^{a}\mathbf{T}_{c} = \left({^{a}\mathbf{T}_{b}}\right)\left({^{b}\mathbf{T}_{c}}\right)$
  2. $^{a}\mathbf{T}_{c} = \begin{bmatrix} {^{a}\mathbf{R}_{b}} & {^{a}\mathbf{v}_{b}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {^{b}\mathbf{R}_{c}} & {^{b}\mathbf{v}_{c}} \\ 0 & 1 \end{bmatrix}$
  3. $^{a}\mathbf{T}_{c} = \begin{bmatrix} \left({^{a}\mathbf{R}_{b}}\right)\left({^{b}\mathbf{R}_{c}}\right) & \left({^{a}\mathbf{R}_{b}}\right)\left({^{b}\mathbf{v}_{c}}\right) + {^{a}\mathbf{v}_{b}} \\ 0 & 1 \end{bmatrix}$

Denavit-Hartenberg (DH) Parameters



https://commons.wikimedia.org/wiki/File:Classic-DHparameters.png

Computationally inefficient and analytically awkward.

What about joints and links?

- Not part of equations per se
- Varying Transforms:
  Revolute Joint: ${}^{i}\mathcal{S}_{i+1}(\theta) = \exp\left(\frac{\theta}{2}{}^{i}\hat{\boldsymbol{u}}_{i+1}\right) + \left(\frac{1}{2}\left({}^{i}v_{i+1}\right) \otimes \exp\left(\frac{\theta}{2}{}^{i}\hat{\boldsymbol{u}}_{i+1}\right)\right)\boldsymbol{\varepsilon}$
  Prismatic Joint: ${}^{j}\mathcal{S}_{j+1}(\ell) = {}^{j}\hbar_{j+1} + \left(\frac{\ell}{2}\left({}^{j}\hat{\boldsymbol{u}}_{j+1}\right) \otimes {}^{j}\hbar_{j+1}\right)\boldsymbol{\varepsilon}$
- Fixed Transforms: ${}^{k}\mathcal{S}_{k+1}$
- 3D Meshes: sets of faces/triangles

Computational Issues

	Storage	Chain Transforms	Transform Point
Quaternion + Vector	7 elements	31 mul., 30 add.	15 mul., 18 add.
Dual Quaternion	8 elements	48 mul., 40 add.	24 mul., 21 add.
Transformation Matrix	12 elements	36 mul., 27 add.	9 mul., 9 add.

Singularities may appear in ln, exp, etc. Usually defined in the limit / can use Taylor series.

Which Representation Should I Use?

Analysis: Dual Quaternion and/or Matrix

- Linear operations

Chaining: Quaternion + Vector

- Fewest operations to chain
- Numerically stable / easy to normalize

Transforming: Matrix

- Fewest operations to transform

Filtering / Estimation: Quaternion + Vector or Dual Quaternion

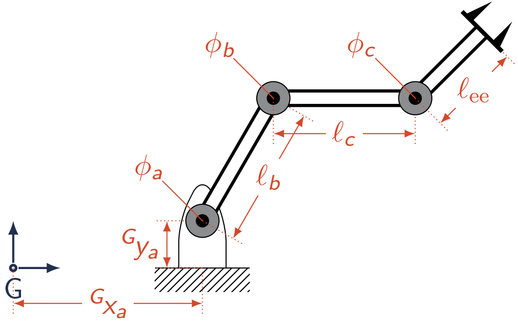
- Numerically stable / easy to normalize

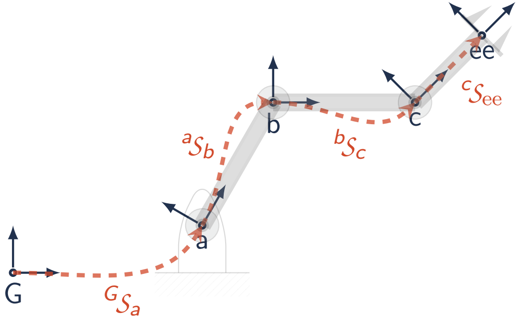
Humans: Axis-Angle and/or Euler Angles

- Easier to visualize angles than sin/cos

Outline

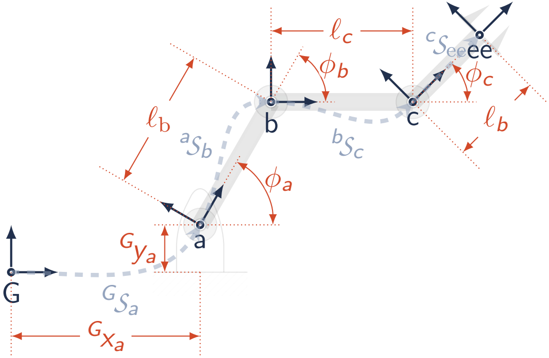
Serial Manipulator





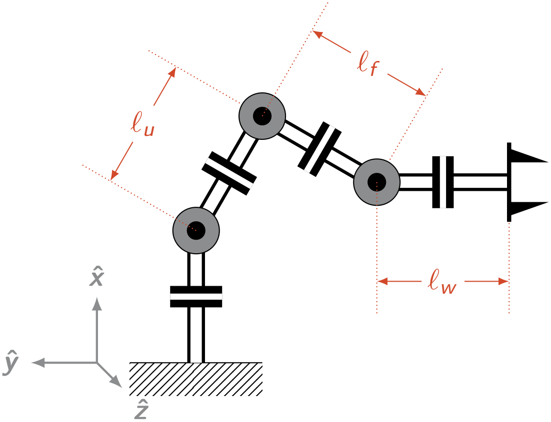
Serial Manipulator

Transforms



- **Relative:** $\mathcal{S} = \mathcal{h} + \frac{1}{2}\vec{v} \otimes \mathcal{h}\varepsilon$
  - ${}^{G}\mathcal{S}_{a} = \exp\left(\frac{\phi_a}{2}\hat{\boldsymbol{k}}\right) + \frac{1}{2}{}^{G}v_a \otimes \exp\left(\frac{\phi_a}{2}\hat{\boldsymbol{k}}\right)\varepsilon$
  - ${}^{a}\mathcal{S}_{b} = \exp\left(\frac{\phi_b}{2}\hat{\boldsymbol{k}}\right) + \frac{1}{2}\ell_b\hat{\boldsymbol{\imath}} \otimes \exp\left(\frac{\phi_b}{2}\hat{\boldsymbol{k}}\right)\varepsilon$
  - ${}^{b}\mathcal{S}_{c} = \exp\left(\frac{\phi_c}{2}\hat{\boldsymbol{k}}\right) + \frac{1}{2}\ell_c\hat{\boldsymbol{\imath}} \otimes \exp\left(\frac{\phi_c}{2}\hat{\boldsymbol{k}}\right)\varepsilon$
  - ${}^{c}\mathcal{S}_{\mathrm{ee}} = 1 + \frac{1}{2}\ell_{\mathrm{ee}}\hat{\boldsymbol{\imath}}\varepsilon$
- **Absolute:** ${}^{G}\mathcal{S}_{n} = {}^{G}\mathcal{S}_{m} \otimes {}^{m}\mathcal{S}_{n}$
  - ${}^{G}\mathcal{S}_{b} = {}^{G}\mathcal{S}_{a} \otimes {}^{a}\mathcal{S}_{b}$
  - ${}^{G}\mathcal{S}_{c} = {}^{G}\mathcal{S}_{b} \otimes {}^{b}\mathcal{S}_{c}$
  - ${}^{G}\mathcal{S}_{\mathrm{ee}} = {}^{G}\mathcal{S}_{c} \otimes {}^{c}\mathcal{S}_{\mathrm{ee}}$

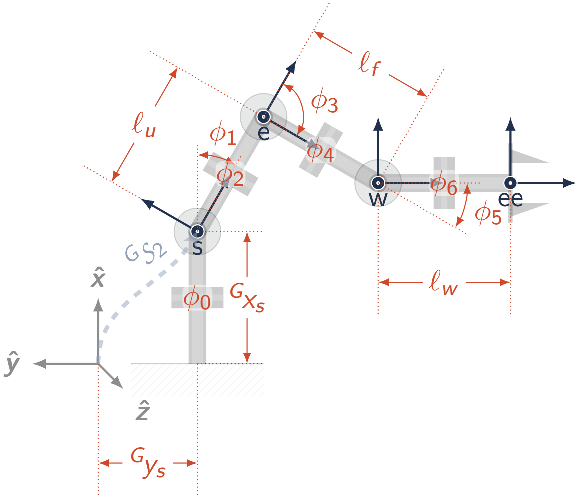
Schunk LWA4D



LWA4D Video

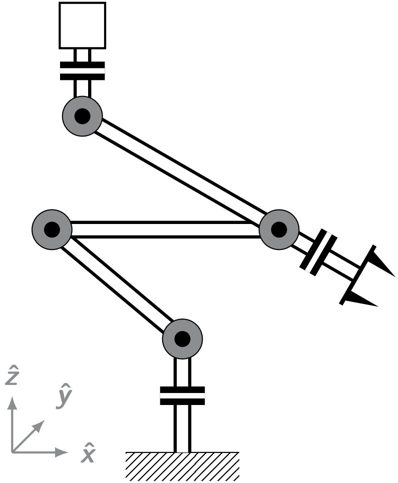
Schunk LWA4D

Transforms



Packbot

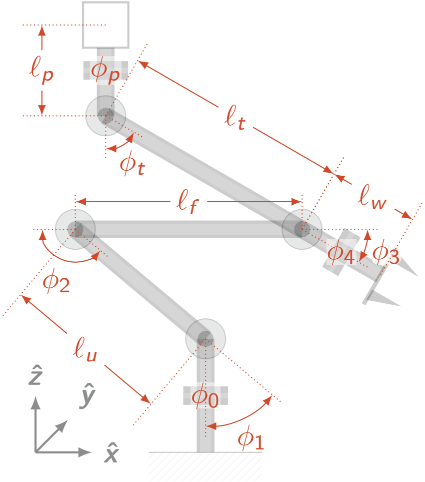
http://endeavorrobotics.com/products



Packbot Video

Packbot

Transforms



Implementation Notes

Fixed Frame: ${}^{\text{parent}}\mathcal{S}_{\text{id}} = \mathcal{S}$

- parent, id: Label
- transform $\mathcal{S}$

Revolute Frame: ${}^{\text{parent}}\mathcal{S}_{\text{id}}(\theta) = \exp\left(\frac{\theta\hat{\boldsymbol{u}}}{2}\right) + \left(\frac{1}{2}\exp\left(\frac{\theta\hat{\boldsymbol{u}}}{2}\right) \otimes \vec{v}\right)\boldsymbol{\varepsilon}$

- parent, id: Label
- axis of rotation ($\hat{\boldsymbol{u}}$)
- fixed translation ($\vec{v}$)

Prismatic Frame: ${}^{\text{parent}}\mathcal{S}_{\text{id}}(\ell) = \mathcal{h} + \left(\frac{1}{2}\ell\hat{\boldsymbol{u}} \otimes \mathcal{h}\right)\boldsymbol{\varepsilon}$

- parent, id: Label
- fixed rotation ($\mathcal{h}$)
- axis of translation ($\hat{\boldsymbol{u}}$)

Scene/Robot: A set of frames

Summary

Local Frames

Dual Quaternions

Other Representations

Kinematic Chains and Trees